

# Weak asymptotic of shock wave formation process

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## Abstract

We construct an asymptotic (in a weak sense) solution corresponding to the shock wave formation in a special situation.

## 1 Introduction

We consider the problem of shock wave formation for the following Hopf type equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \quad (1)$$

where we assume that  $f \in C^3$  and the inequality  $f''(u) > 0$  holds on the range of the solution  $u$ . We shall consider the special initial condition for Eq. (1):

$$u|_{t=0} = u_0^0 + (u_1(x) - u_0^0)H(a_1 - x) + (U - u_1(x))H(a_2 - x), \quad (2)$$

where  $u_0^0$ ,  $U$ , and  $a_1 > a_2$  are constants,  $H$  is the Heaviside function, the function  $u_1(x)$  is determined by the equation

$$f'(u_1(x)) = -Kx + b, \quad K, b = \text{const}, \quad (3)$$

and, in addition, we assume that  $u_1(a_1) = u_0^0$  and  $u_1(a_2) = U$ .

Such a function appears<sup>1</sup> in the construction of the entropy solution to the Cauchy problem with an "unstable" initial jump.

It follows from the choice of such an initial condition that the approximation of problem (1)–(2) (a weak asymptotic solution) for all  $t$  is an element of the asymptotic subalgebra

$$\mathcal{B}\{1, H_1(x - \varphi_1, \varepsilon), H_2(x - \varphi_2, \varepsilon)\}$$

introduced in [1].

Roughly speaking, this means that at any time moment the weak limit of the weak asymptotic solution is a linear combination of the Heaviside functions

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<sup>1</sup>E. Yu. Panov drew the author's attention to this fact.

$H(x - \varphi_1)$  and  $H(x - \varphi_2)$  with smooth in  $t$  ( $\varepsilon > 0$ ) coefficients and that there are no additional jumps. In turn, this means that at time

$$t^* = \frac{a_1 - a_2}{f'(U) - f'(u_0^0)}$$

all characteristics meet at the same point  $x^* = a_i + V_i t^*$ ,  $V_1 = f'(u_0^0)$ ,  $V_2 = f'(U)$ ,  $i = 1, 2$ .

More precisely, for  $0 < t < t^*$ , the solution of problem (1)–(2) is given by the formula

$$u = u_0^0 + (u_1(x_0(x, t)) - u_0^0)H(\varphi_1 - x) + (U - u_1(x_0(x, t)))H(\varphi_2 - x), \quad (4)$$

where the function  $u_1(x_0(x, t))$  has the form

$$u_1(x_0(x, t)) = u_1\left(a_i + \frac{x - \varphi_i(t)}{\psi_0}\psi_0^0\right) = u_1\left(\frac{x - bt}{1 - Kt}\right).$$

Here  $\varphi_i(t) = a_i + V_i t$ ,  $i = 1, 2$ ,  $\psi_0 = \varphi_1(t) - \varphi_2(t)$ , and  $\psi_0^0 = a_1 - a_2$ .

For  $t = t^*$  the plot of the function  $u = u(x, t)$  is the graph

$$((-\infty, x^*), U) \cup (x^*, (U, u_0^0)) \cup ((x^*, \infty), u_0^0).$$

We note that if we set

$$\bar{u}(x, t) = \begin{cases} u_1(x_0(x, t)), & t < t^*, \quad t > t^*, \\ u_0^0, & t = t^*, \quad x < x^*, \\ U, & t = t^*, \quad x > x^*, \\ \bar{u} \in [u_0^0, U], & t = t^*, \quad x = x^*, \end{cases}$$

then the function  $\bar{u}(x, t)$  is defined for all values of  $t$  and, for  $t < t^*$ , is a solution of Eq. (1) satisfying the initial condition (2) for  $t = 0$ . Our goal is to "correct" the function  $\bar{u}(x, t)$  and to obtain an analytic formula that, for  $t < t^*$ , determines a function close to  $\bar{u}(x, t)$  and, for  $t > t^*$ , a function close to the function

$$u = u_0^0 + H(c(t - t^*) - (x - x^*))U, \quad (5)$$

where

$$c = \left. \frac{[f(u)]}{[u]} \right|_{x=ct} = \frac{f(U) - f(u_0^0)}{U - u_0^0}.$$

The answer is given by formula (10) below.

We note that the function  $u$  determined by relation (4) for  $t < t^*$  is continuous everywhere except the points lying on the curves  $x = \varphi_i(t)$ ,  $i = 1, 2$ , and, at points of these curves, the function has weak discontinuities (the derivatives of the function have jumps at these points). Therefore, the formation of the shock wave (5) from (4) can be treated as the result of interaction (confluence) of weak discontinuities. Moreover, for  $t < t^*$ , although the derivatives are discontinuous, the solution of such problems (that is continuous, but with jumps

of the derivatives on some smooth nonintersecting curves) can be constructed by the method of characteristics.

We note that, in the case  $f(u) = u^2$ , the problem of constructing the global asymptotic solution of problem (1), (2) was solved as an example in [2]. The asymptotic solution constructed in [2] is a weak asymptotic solution. We recall how it is determined. By  $O_{\mathcal{D}'}(\varepsilon^\alpha)$  we denote generalized functions that, in general, depend on the parameters  $t$  and  $\varepsilon$  and are such that for any test function  $\eta(x)$ , the estimate

$$\langle O_{\mathcal{D}'}(\varepsilon^\alpha), \eta(x) \rangle = O(\varepsilon^\alpha)$$

holds, where the estimate on the right-hand side is understood in the usual sense and locally uniform in  $t$ , i.e.,  $|O(\varepsilon^\alpha)| \leq C_T \varepsilon^\alpha$  for  $t \in [0, T]$ .

A function  $u_\varepsilon = u_\varepsilon(x, t)$  is called a weak asymptotic solution of problem (1), (2) if

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial f(u_\varepsilon)}{\partial x} = O_{\mathcal{D}'}(\varepsilon), \quad u_\varepsilon \Big|_{t=0} - u \Big|_{t=0} = O_{\mathcal{D}'}(\varepsilon). \quad (6)$$

The goal of this paper is to construct such a function in the case of a general convex nonlinearity  $f(u)$ . This is achieved in Sections 2 and 3.

In Section 4 we introduce auxiliary formulas and statements of the weak asymptotic method.

We note that if the solution  $u_\varepsilon$  satisfies the Oleinik–Kruzhkov stability conditions [3, 4], then it follows from (6) that  $u_\varepsilon$  differs from  $u$  by a measure [5] whose values are estimated as  $O(\varepsilon)$ . Indeed, it is easy to verify that the right-hand sides in (6) arising in our construction belong to  $C([0, T], L^1(\mathbb{R}_x^1))$  and can be estimated as  $O(\varepsilon)$  in the sense of the  $L^1$ -norm. Therefore, according to the results in [3, 4],  $u_\varepsilon$  is an asymptotic of the solution to the Cauchy problem (1)–(2) in  $L^1$ . This is done in Section 5.

We also note that the asymptotic (in the usual weak sense) solution describing the global behavior of the solution of the Cauchy problem with a small viscosity and a smooth initial condition for the equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}$$

was first constructed by A. M. Il'in [6]. This was an important achievement in the asymptotic theory.

In contrast to our paper, in A. M. Il'in's paper an arbitrary smooth initial condition was considered. In [2], in the case  $f(u) = u^2$ , it was explained how the solution constructed there can be used to obtain the global weak asymptotic for a more general Cauchy problem. For this, it was proposed to consider an interpolation of the initial function by linear splines. Also, we can use different approach. For given smooth initial data  $u_0(x)$ ,  $x \in \mathbf{R}$ , we can find (assume finite) set of points  $x_k^0 \in \mathbf{R}$ ,  $k \in 1, \dots, N$ ,  $N \in \mathbf{N}$ , which reaches the point of the gradient catastrophe in the moment  $t = t_k$ . Now, instead of given initial data  $u_0(x)$ ,  $x \in \mathbf{R}$ , we impose initial data  $u_{0\varepsilon}(x)$ ,  $x \in \mathbf{R}$ , which differs from the function  $u_0(x)$ ,  $x \in \mathbf{R}$ , in the intervals  $[x_k^0 - \varepsilon^\mu, x_k^0 + \varepsilon^\mu]$ ,  $0 < \mu < 1$ ,  $k \in 1, \dots, N$ .

In those intervals the function  $u_{0\varepsilon}(x)$ ,  $x \in \mathbf{R}$ , has form (10). It is obvious that we have

$$\|u_0(x) - u_{0\varepsilon}(x)\|_{L^1(\mathbf{R})} = \mathcal{O}(\varepsilon^\mu).$$

Then, we solve the Cauchy problem corresponding to new initial data  $u_{0\varepsilon}(x)$ ,  $x \in \mathbf{R}$ , using method of characteristics in a way that for the "inserted" parts (the one in the intervals  $[x_k^0 - \varepsilon^\mu, x_k^0 + \varepsilon^\mu]$ ,  $0 < \mu < 1$ ,  $k \in 1, \dots, N$ ) we use "the new characteristics" given by (11) and for the rest of the function  $u_{0\varepsilon}(x)$  ( $\equiv u_0(x)$  for  $x \notin [x_k^0 - \varepsilon^\mu, x_k^0 + \varepsilon^\mu]$ ) we use ordinary characteristics. This will be the subject of further investigations.

## 2 Description of the formula for the weak asymptotic solution

To construct a weak asymptotic solution describing the passage from (4) to (5), we introduce some auxiliary constructions.

We define a function  $\xi(x_0)$  as a solution of the implicit equation

$$U + u_0^0 = u_1(x_0) + u_1(\xi(x_0)), \quad (7)$$

which is solvable due to (3).

Obviously,  $\xi : [a_2, a_1] \rightarrow [a_2, a_1]$  is a smooth isomorphism and  $\xi(\xi(x_0)) = x_0$ .

We introduce the function  $U_1(x_0, \rho)$ , by setting

$$U_1(x_0, \rho) = B_2(\rho)u_1(x_0) + B_1(\rho)u_1(\xi(x_0)), \quad (8)$$

where the functions  $B_i(\rho)$ ,  $i = 1, 2$ , are defined in Lemma 4.1 and the function  $\rho = \rho(\tau)$  is defined below, see (12), (13) and

$$\tau = \frac{\varphi_{10}(t) - \varphi_{20}(t)}{\varepsilon}, \quad \varphi_{i0}(t) = a_i + f'(u_1(a_i))t, \quad i = 1, 2.$$

Note that, by (7), (8), and the formulas for  $B_i$  at the end of Lemma 4.1, we have

$$U + u_0^0 - U_1(x_0, \rho) = U_1(\xi(x_0), \rho), \quad U_1(x_0, \rho) = u_1(x_0) + O(\rho^{-N}), \quad \rho \rightarrow \infty. \quad (9)$$

We shall seek a weak asymptotic solution of problem (1)–(2) in the form

$$\begin{aligned} u_\varepsilon(x, t) &= u_0^0 + \left( U_1(x_0(x, t, \tau), \rho) - u_0^0 \right) \omega_1 \left( \frac{\phi_1 - x}{\varepsilon} \right) \\ &\quad + \left( U - U_1(x_0(x, t, \tau), \rho) \right) \omega_2 \left( \frac{\phi_2 - x}{\varepsilon} \right), \end{aligned} \quad (10)$$

where  $\omega_i(z) \rightarrow 0, 1$  as  $z \rightarrow \mp\infty$ ,  $\frac{d^\alpha \omega_i}{dz^\alpha} = O(|\tau|^{-N})$ , where  $|z| \rightarrow \infty$ ,  $\alpha > 0$  and  $N > 0$  are arbitrary numbers, and  $\phi_i = \phi_i(t, \varepsilon)$ ,  $i = 1, 2$ ,  $x_0(x, t, \tau)$  are the desired functions.

As noted in the last Section (see Sec. 4.1), the functions  $\omega_i((\phi_i - x)/\varepsilon)$  approximate (in the weak sense) the Heaviside functions  $H(\phi_i - x)$ ,

$$\omega_i\left(\frac{\phi_i - x}{\varepsilon}\right) = H(\phi_i - x) + O_{\mathcal{D}'}(\varepsilon), \quad i = 1, 2.$$

We shall seek the functions  $\phi_i = \phi_i(t, \varepsilon)$ ,  $i = 1, 2$ , in the form

$$\phi_i = \hat{\varphi}_i(t, \tau) + \psi_0 \hat{\phi}(t, \tau), \quad i = 1, 2.$$

Here,  $\hat{\phi}(t, \tau)$  is such that it satisfies  $\hat{\phi}(t, \tau)|_{\tau \rightarrow \infty} = 0$ . Furthermore,  $\hat{\varphi}_i(t, \tau)$  is an analog of the trajectories of weak discontinuities of  $\varphi_i(t)$  in (4). The functions  $\hat{\varphi}_i(t, \tau)$ ,  $i = 1, 2$ , can be found from the equations for the "new characteristics" and, as  $\tau \rightarrow \infty$  (i.e., before the confluence of weak singularities), these functions are close to the trajectories  $\varphi_i(t)$  from the preceding section.

To find the functions  $x_0(x, t, \tau)$ , we introduce the differential equation for the "new characteristics"

$$\frac{dx}{dt} = B_2(\rho)f'(U_1(x_0, \rho)) + B_1(\rho)f'(U_1(\xi(x_0), \rho)) + q(\tau, \rho), \quad x|_{t=0} = x_0. \quad (11)$$

The function  $q(\tau, \rho)$  is assumed to be smooth and to satisfy the estimate

$$|\tau q(\tau, \rho)| \leq \text{const}. \quad (12)$$

Its appearance itself is caused by the fact that the function  $U_1(x, \rho)$ , which replaces the function  $u_1(x_0)$  in formula (4), depends on time (via the function  $\rho = \rho(\tau)$  determined in (13), (14)). Therefore, this function is not preserved along the usual trajectories corresponding to quasilinear equations. The "new trajectories" are just given by Eq. (11), where the function  $\rho$  is determined as follows. By  $x(x_0, t, \tau)$  we denote the solution of (11) and introduce the functions  $\hat{\varphi}_i(t, \tau) = x(a_i, t, \tau)$ ,  $i = 1, 2$ . We set

$$\rho = \frac{\hat{\varphi}_1(t, \tau) - \hat{\varphi}_2(t, \tau)}{\varepsilon} = \frac{\phi_1(t, \tau) - \phi_2(t, \tau)}{\varepsilon}.$$

We note that  $U_1(a_1, \tau) = B_2 u_0^0 + B_1 U$  and  $U_1(a_2, \tau) = B_2 U + B_1 u_0^0$ ; hence from (11) we easily obtain the following equation for  $\rho = \rho(\tau)$ :

$$\frac{d\rho}{d\tau} = (B_2(\rho) - B_1(\rho))(f'(B_2 u_0^0 + B_1 U) - f'(B_2 U + B_1 u_0^0))(\psi'_0)^{-1}. \quad (13)$$

Obviously, by definition,

$$\rho \tau^{-1} \rightarrow 1 \quad \text{as} \quad \tau \rightarrow \infty \quad \text{and} \quad \frac{d\rho}{d\tau} > 0 \quad (\text{since} \quad \psi'_0 < 0). \quad (14)$$

We denote the right-hand side of (13) by  $G(\rho)$ . Obviously,  $G(\rho_0) = 0$ , where  $\rho_0$  is a number such that  $B_1(\rho_0) = B_2(\rho_0)$ , and hence (see Lemma 4.1)

$$B_1(\rho_0) = B_2(\rho_0) = 1/2. \quad (15)$$

We assume that  $\rho_0 > 0$ . This is a condition imposed on  $B_j$ . It is easy to verify that  $\frac{dG}{d\rho} \Big|_{\rho=\rho_0} = 0$ , while

$$\frac{d^2G}{d\rho^2} \Big|_{\rho=\rho_0} = -8B_{2\rho}^{\prime 2}(U - u_0^0)f''\left(\frac{U + u_0^0}{2}\right) \neq 0. \quad (16)$$

It follows from Eq. (13) and inequality (16) that the relations

$$\rho \rightarrow \rho_0 + O(1/|\tau|), \quad \dot{\rho} = O(1/|\tau|^2) \quad (17)$$

hold as  $\tau \rightarrow -\infty$ .

Thus, independently of (11), the function  $\rho = \rho(\tau)$  is defined as a solution of problem (13), (14). Therefore, the function  $x(x_0, t, \tau)$  from (11) is also defined. Now put

$$\hat{x}(x_0, t, \tau) = X(x_0, t) + \psi_0 X_1(x_0, \tau), \quad (18)$$

where

$$X(x_0, t) = x_0 + f'(u_0(x_0))t = x_0\psi_0(\psi_0^0)^{-1} + bt.$$

Inserting  $\hat{x}$  in (11) instead of  $x$  we have

$$\begin{aligned} X_1 &= \frac{1}{\psi_0' \tau} \int_0^\tau \left[ B_2(\rho) f'(U_1(x_0, \rho)) + B_1(\rho) f'(U_1(\xi(x_0), \rho)) \right. \\ &\quad \left. + q(\tau', \rho) - f'(u_0(x_0)) \right] d\tau' \\ &= \frac{1}{\psi_0' \tau} \int_0^\tau \left[ B_2(\rho) f'(U_1(x_0, \rho)) + B_1(\rho) f'(U_1(\xi(x_0), \rho)) \right. \\ &\quad \left. + q(\tau', \rho) \right] d\tau' - (\psi_0^0)^{-1}x_0 - b(\psi_0')^{-1}. \end{aligned} \quad (19)$$

It is easy to verify that the following representation is true:

$$\hat{x} = x^* + \frac{\psi_0}{\psi_0' \tau} \int_0^\tau \left[ B_2(\rho) f'(U_1(x_0, \rho)) + B_1(\rho) f'(U_1(\xi(x_0), \rho)) + q(\tau', \rho) \right] d\tau',$$

which follows from the identity

$$x_0\psi_0(\psi_0^0)^{-1} + bt - \psi_0(\psi_0^0)^{-1}x_0 - \psi_0 b(\psi_0')^{-1} = -b \frac{\psi_0^0}{\psi_0'} = x^*.$$

It is not difficult to see that the solution  $\hat{x}$  given by formula (19) is not the exact solution of (11). Actually, for  $t = 0$  (i.e. for  $\tau \rightarrow +\infty$ ) from (19) we obtain

$$(X_0 + \psi_0 X_1)|_{t=0} = x_0 + \mathcal{O}(\varepsilon).$$

Obviously, for  $t \in [0, T]$ ,  $T \in \mathbf{R}$ , we have:

$$x(x_0, t, \tau) = \hat{x}(x_0, t, \tau) + O(\varepsilon).$$

It is easy to verify that the term  $O(\varepsilon)$  in the last relation has the form

$$O(\varepsilon) = \psi_0 X_1 \Big|_{t=0} = \frac{\psi_0}{\psi'_0 \tau} \int_0^\infty \left[ B_2(\rho) f'(U_1(x_0, \rho)) + B_1(\rho) f'(U_1(\xi(x_0), \rho)) \right. \\ \left. + q(\tau', \rho) - f'(u_0(x_0)) \right] d\tau' \stackrel{\text{def}}{=} \varepsilon g(x_0).$$

Finally, we obtain

$$x(x_0, t, \tau) = \hat{x}(x_0, t, \tau) + \varepsilon g(x_0).$$

Let us calculate the derivative  $\frac{\partial \hat{x}}{\partial x_0}$ . By (18), (19), we have

$$\begin{aligned} \frac{\partial x}{\partial x_0} &= \frac{\psi_0}{\psi'_0 \tau} \int_0^\tau \frac{\partial}{\partial x_0} \left[ B_2(\rho) f'(U_1(x_0, \rho)) + B_1(\rho) f'(U_1(\xi(x_0), \rho)) \right] d\tau \\ &= \frac{\psi_0}{\psi'_0 \tau} \int_0^\tau \left[ B_2(\rho) f''(U_1(x_0, \rho)) - B_1(\rho) f''(U_1(\xi(x_0), \rho)) \right] \frac{\partial U_1}{\partial x_0}(x_0, \rho) d\tau \end{aligned} \quad (20)$$

Here we used the relation

$$\frac{\partial U_1}{\partial x_0}(x_0, \rho) = -\frac{\partial U_1}{\partial x_0}(\xi(x_0), \rho),$$

which follows from the definition of the function  $U_1(x_0, \rho)$  in (9).

We agree that the symbol  $\sim$  denotes the following equivalence relation

$$f \sim g \leftrightarrow \lim \frac{f}{g} = \text{const} \neq 0. \quad (21)$$

Then, as  $\tau \rightarrow -\infty$ , we have

$$\begin{aligned} \frac{\partial U_1}{\partial x_0} &\sim B_1 - \frac{1}{2} \sim \frac{1}{\tau}, \quad B_2 - \frac{1}{2} \sim \frac{1}{\tau}, \\ U_1(x_0, \rho) &\rightarrow \frac{U + u_0^0}{2}, \quad U_1(\xi(x_0), \rho) \rightarrow \frac{U + u_0^0}{2}. \end{aligned}$$

Therefore,

$$\left[ B_2(\rho) f''(U_1(x_0, \rho)) - B_1(\rho) f''(U_1(\xi(x_0), \rho)) \right] \sim \frac{1}{\tau}.$$

Hence the integral in (20) converges as  $\tau \rightarrow -\infty$  and

$$\frac{\partial \hat{x}}{\partial x_0} \sim \frac{\psi_0}{\psi'_0 \tau}, \quad \tau \rightarrow -\infty.$$

As  $\tau \rightarrow \infty$ , we have  $U_1(x_0, \rho) \rightarrow u_1(x_0)$  (since  $B_2 \rightarrow 1$ ) and the integrand in (20) tends to the limit

$$f''(u_1(x_0)) \frac{\partial u_1}{\partial x_0}(x_0) = \frac{\psi'_0}{\psi_0^0}.$$

Thus

$$\frac{\partial \hat{x}}{\partial x_0} \rightarrow \frac{\psi_0}{\psi_0^0}, \quad \tau \rightarrow \infty.$$

We note that the solvability of the equation  $x(x_0, t, x) = x$  with respect to  $x_0$  globally in  $t$  can hardly be ignored.

In our constructions, we shall hence use the following approximate expression for the solution of Eq. (11), namely,

$$x(x_0, t, \tau) = \hat{x}(x_0, t, \tau) + \varepsilon(g(x_0) + Ax_0),$$

where  $A > 0$ ,  $A = \text{const}$ . Clearly, we have

$$x|_{t=0} = x_0 + \varepsilon Ax_0,$$

and hence  $x_0(x, t, \tau)|_{t=0} = x - \varepsilon Ax + O(\varepsilon)$ .

As is easy to verify, this means that, in the sense of  $O_{D'}$ -estimates, the initial condition in (???) will be satisfied with accuracy up to  $O_{D'}(\varepsilon)$ . We prove that the constant  $A$  can be chosen so that the inequality

$$\frac{\partial x}{\partial x_0} > 0$$

holds uniformly in  $t$ .

We have

$$\begin{aligned} \frac{\partial x}{\partial x_0} &= 1 - kt + \frac{\psi_0}{\psi_0' \tau} \int_0^\tau \left[ B_2 f''(U_1(x_0, \rho)) - B_1 f''(U_1(\xi(x_0), \rho)) \right. \\ &\quad \left. - f''(u_1(x_0)) \frac{\partial u}{\partial x_0} \right] d\tau' + \varepsilon g'(x_0) + \varepsilon A. \end{aligned}$$

Recall that  $t^* = k^{-1}$ ,  $\psi_0(t^*) = 0$ ,  $\tau = \psi_0(t)/\varepsilon$ . Hence for  $t \leq t^*$ , by Lemma 4.2, we have the estimate

$$\frac{\psi_0}{\psi_0' \tau} \int_0^\tau \left[ B_2 f''(U_1(x_0, \rho)) - B_1 f''(U_1(\xi(x_0), \rho)) - f''(u_1(x_0)) \frac{\partial u_1}{\partial x_0} \right] d\tau' = O(\varepsilon)$$

Similarly, for  $t \geq t^*$ , we have

$$1 - kt + \frac{\psi_0}{\psi_0' \tau} \int_0^\infty \left[ B_2 f''(U_1(x_0, \rho)) - B_1 f''(U_1(\xi(x_0), \rho)) - f''(u_1(x_0)) \frac{\partial u_1}{\partial x_0} \right] d\tau' = O(\varepsilon).$$

It follows from these estimates that there is a possibility to choose the constant  $A$ . Thus the equation

$$X_0(x_0, t) + \psi_0 X_1(x_0, t, \varepsilon) + \varepsilon(g(x_0) + Ax_0) = x \tag{22}$$

can be globally solved with respect to  $x_0$ .

In this case, the derivatives of the exact solution of Eq. (11) differ from the function in the right-hand side of (22) and from the function  $\hat{x}(x_0, t, \tau)$  by  $O(\varepsilon)$ . Therefore, in what follows, to simplify the calculations, we shall use all these functions.

### 3 Construction of the weak asymptotic solution

We substitute the function  $u_\varepsilon(x, t)$  into Eq. (1). Using Lemma 4.1 and the formula for weak asymptotic of the approximations in Sec. 4, we obtain

$$\begin{aligned}
\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u_\varepsilon) &= \phi_{1t}(U_1(x_0(\phi_1, t, \tau), \rho) - u_0^0)\delta(x - \phi_1) \\
&\quad + \phi_{2t}(U - U_1(x_0(\phi_2, t, \tau), \rho))\delta(x - \phi_2) \\
&\quad + \frac{\partial U_1}{\partial x_0} \frac{\partial x_0}{\partial t} [H(\phi_1 - x) - H(\phi_2 - x)] \\
&\quad + \frac{\partial U_1}{\partial x_0} \frac{\partial x_0}{\partial x} [B_2(\rho) f'(U_1(x_0(x, t, \tau), \rho)) \\
&\quad + B_1(\rho) f'(U_1(\xi(x_0(x, t, \tau)), \rho))] [H(\phi_1 - x) - H(\phi_2 - x)] \\
&\quad - \delta(x - \phi_1) \left[ B_2(\rho) \left( f(U_1(x_0(\phi_1, t, \tau), \rho)) - f(u_0^0) \right) \right. \\
&\quad \left. + B_1(\rho) \left( f(U) - f(U_1(\xi(x_0(\phi_1, t, \tau)), \rho)) \right) \right] \\
&\quad - \delta(x - \phi_2) \left[ B_1(\rho) \left( f(U_1(\xi(x_0(\phi_2, t, \tau)), \rho)) - f(u_0^0) \right) \right. \\
&\quad \left. + B_2(\rho) \left( f(U) - f(U_1(x_0(\phi_2, t, \tau), \rho)) \right) \right] \\
&\quad + \frac{\partial U_1}{\partial t}(x_0, \rho) \Big|_{x_0=x_0(x, t, \tau)} [H(\phi_1 - x) - H(\phi_2 - x)] + O_{D'}(\varepsilon).
\end{aligned} \tag{23}$$

Although there are rather many terms on the right-hand side, it is easy to understand this formula. The terms containing the factors  $(H(\phi_1 - x) - H(\phi_2 - x))$  correspond to the substitution into the equation of the function  $u_\varepsilon$  determined in (10) between the points  $x = \phi_i$  and with Lemma 4.1 taken into account.

The terms containing the delta functions, i.e., the factors  $\delta(x - \phi_i)$ ,  $i = 1, 2$ , appear due to the fact that  $U(x_0(\phi_i(t, \tau), \rho)) \neq u_0^0$  and  $U(x_0(\phi_i(t, \tau), \rho)) \neq U$ , but as  $\tau \rightarrow \infty$  (i.e., before the interaction) we have  $\rho \sim \tau$  (see (14)) and hence, for any  $N > 0$ , we have

$$U(x_0(\phi_1(t, \tau), \rho)) - u_0^0 = O(\varepsilon^N), \quad U(x_0(\phi_2(t, \tau), \rho)) - U = O(\varepsilon^N).$$

We start analyzing the terms in (22) from the last one (which has the estimate  $O(\varepsilon^{-1})$  in the  $C$ -norm):

$$\begin{aligned}
\frac{\partial U_1}{\partial t} &= \varepsilon^{-1} \psi'_0 \dot{\rho} [B'_{2\rho} u_1(x_0) + B'_{1\rho} u_1(\xi(x_0))] \\
&= \varepsilon^{-1} \psi'_0 \dot{\rho} B'_{2\rho} [U + u_0^0 - 2u_1(x_0(x, t, \tau))].
\end{aligned} \tag{24}$$

Applying the Taylor formula at the points  $x = \phi_1$  and  $x = \phi_2$ , for any test

function  $\eta(x)$ , we obtain

$$\begin{aligned} & \varepsilon^{-1} \psi'_0 \dot{\rho} B'_2 \int_{\phi_2}^{\phi_1} [U + u_0^0] \eta(x) dx \\ &= \frac{\psi'_0}{2} \rho \dot{\rho} B'_{2\rho} (U + u_0^0) \langle \{\delta(x - \phi_1) + \delta(x - \phi_2)\}, \eta(x) \rangle + O(\varepsilon), \end{aligned} \quad (25)$$

and  $B'_{2\rho} = O(|\rho|^{-N})$  for any  $N > 0$  as  $\rho \rightarrow \infty$ ,  $B'_{2\rho} \rightarrow \text{const}$  as  $\rho \rightarrow \rho_0$  ( $\tau \rightarrow -\infty$ ), and  $\dot{\rho} = O(|\tau|^{-2})$  as  $\tau \rightarrow -\infty$  (see (17)).

Let us consider the remaining term. We have

$$\int_{\phi_2}^{\phi_1} u_1(x_0(x, t, \tau)) \eta(x) dx = \int_{a_2}^{a_1} u_1(x_0) \eta(x(x_0, t, \tau) + \psi_0 \hat{\phi}) \frac{dx}{dx_0} dx_0. \quad (26)$$

Let us note that (see (18))

$$\frac{\partial x}{\partial x_0} = \psi_0 \left( (\psi_0^0)^{-1} + \frac{\partial X_1}{\partial x_0} \right) = \varepsilon \tau \left( (\psi_0^0)^{-1} + \frac{\partial X_1}{\partial x_0} \right). \quad (27)$$

Hence the right-hand side in (23) is bounded in the weak sense as  $\varepsilon \rightarrow 0$ . We now note that the following relations hold:

$$\begin{aligned} \eta(x(x_0, t, \tau) + \psi_0 \hat{\phi}) &= \eta(x(a_i, t, \tau) + \psi_0 \hat{\phi}) + \eta'_x \frac{\partial x}{\partial x_0} \Big|_{x_0=c_i}, \\ c_i &\in (a_i, x_0), \quad i = 1, 2. \end{aligned}$$

Recalling that  $x(a_i, t, \tau) = \hat{\varphi}_i(t, \tau)$ ,  $i = 1, 2$ , and again using (26), (14) and (17) we obtain

$$\begin{aligned} & \varepsilon^{-1} \psi'_0 \dot{\rho} B'_2 \int_{\phi_2}^{\phi_1} u_1(x_0(x, t, \tau)) \eta(x) dx \\ &= \frac{1}{2} \langle \delta(x - \phi_1) + \delta(x - \phi_2), \eta(x) \rangle \psi'_0 \dot{\rho} \tau B'_2 \\ & \quad \times \int_{a_2}^{a_1} u_1(x_0) \left( (\psi_0^0)^{-1} + \frac{\partial X_1}{\partial x_0} \right) dx_0 + O(\varepsilon). \end{aligned}$$

Finally, we have

$$\frac{\partial U_1}{\partial t} [H(\phi_1 - x) - H(\phi_2 - x)] = g(\tau, \rho) (\delta(x - \phi_1) + \delta(x - \phi_2)) + O_{\mathcal{D}'}(\varepsilon),$$

where

$$g(\tau, \rho) = \frac{\psi'_0}{2} \rho \dot{\rho} B'_{2\rho} (U + u_0^0) - \psi'_0 \dot{\rho} \tau B'_{2\rho} \int_{a_2}^{a_1} u_1(x_0) \left( (\psi_0^0)^{-1} + \frac{\partial X_1}{\partial x_0} \right) dx_0. \quad (28)$$

It is easy to see that, by formula (17), we have the estimate

$$|\tau^2 g(\tau, \rho)| \leq \text{const.}$$

Moreover, the function  $g(\tau, \rho)$  is integrable, and the integral  $\int_0^\tau g(\tau, \rho) d\tau$  converges. Indeed, the integral of the first term converges because of the estimates given after formula (24), and the integral of the second term, in its properties, coincides with the last integral in formula (20).

Now we consider the remaining terms that contain the difference  $H(\phi_1 - x) - H(\phi_2 - x)$  as the multiplier. For any function  $\eta(x) \in C_0^\infty$ , taking into account the relation

$$\frac{\partial x_0}{\partial t} = -\frac{\partial x_0}{\partial x} \frac{\partial x}{\partial t},$$

we have

$$\begin{aligned} & \left\langle \left[ \frac{\partial U_1}{\partial x_0} \frac{\partial x_0}{\partial t} + \frac{\partial U_1}{\partial x_0} \frac{\partial x_0}{\partial x} \left( B_2(\rho) f'(U_1(x_0(x, t, \tau), \rho)) \right. \right. \right. \\ & \quad \left. \left. \left. + B'_1(\rho) f'(U + u_0^0 - U_1(x_0(x, t, \tau), \rho)) \right) \right] (H(\phi_1 - x) - H(\phi_2 - x)), \eta(x) \right\rangle \\ &= \int_{\phi_2}^{\phi_1} \frac{\partial U_1}{\partial x_0} \frac{\partial x_0}{\partial x} \left[ -\frac{\partial x}{\partial t} + B_2(\rho) f'(U_1(x_0(x, t, \tau), \rho)) \right. \\ & \quad \left. \left. + B_1(\rho) f'(U_1(\xi(x_0(x, t, \tau)), \rho)) \right] \eta(x) dx. \end{aligned} \quad (29)$$

By (11), the expression in square brackets on the right-hand side of (28) is just  $q(\tau, \rho)$ .

We consider the integral

$$\int_{\phi_2}^{\phi_1} \frac{\partial U_1}{\partial x_0} \frac{\partial x_0}{\partial x} \eta(x) dx$$

and pass to the variables  $x_0$  precisely as in (25). We obtain

$$\int_{\phi_2}^{\phi_1} \frac{\partial U_1}{\partial x_0} \frac{\partial x_0}{\partial x} \eta(x) dx = \int_{a_2}^{a_1} \frac{\partial U_1}{\partial x_0} \eta(x(x_0, t, \tau) + \psi_0 \hat{\phi}) dx_0.$$

Recall that

$$\frac{\partial U_1}{\partial x_0} \sim \frac{1}{\tau}, \quad \tau \rightarrow -\infty, \quad \frac{\partial U_1}{\partial x_0} \rightarrow \frac{\partial u_1(x_0)}{\partial x_0}, \quad \tau \rightarrow \infty.$$

From the conjectural estimate (12) for the function  $q(\tau, \rho)$ , using the Taylor formula as in (24), we obtain

$$q \int_{\phi_2}^{\phi_1} \frac{\partial U_1}{\partial x_0} \frac{\partial x_0}{\partial x} \eta(x) dx = \frac{q}{2} \int_{a_2}^{a_1} \frac{\partial U_1}{\partial x_0} dx_0 \langle \delta(x - \phi_1) + \delta(x - \phi_2), \eta(x) \rangle + O(\varepsilon).$$

Taking into account the definition of the function  $U_1(x_0, \rho)$ , we can easily calculate the integral on the right-hand side of the last formula and obtain

$$\begin{aligned} & q \int_{\phi_2}^{\phi_1} \frac{\partial U_1}{\partial x_0} \frac{\partial x_0}{\partial x} \eta(x) dx \\ &= \frac{q(B_2 - B_1)(u_0^0 - U)}{2} \langle \delta(x - \phi_1) + \delta(x - \phi_2), \eta(x) \rangle + O(\varepsilon). \end{aligned} \quad (30)$$

We choose the function  $q(\tau, \rho)$  so that the following relation hold:

$$q(B_2 - B_1)(u_0^0 - U) = -g(\tau, \rho). \quad (31)$$

Obviously, we have

$$\begin{aligned} q(\tau, \rho) &\sim g(\tau, \rho) = O(\tau^{-N}) \quad \forall N, \quad \tau \rightarrow \infty; \\ q(\tau, \rho) &\sim \tau g(\tau, \rho) \sim \frac{1}{\tau}, \quad \tau \rightarrow -\infty. \end{aligned}$$

Hence the estimate (12) holds and our constructions that lead to (29) are well defined.

It is left to obtain the function  $\hat{\phi}$  appearing in the definition of the functions  $\phi_{it}$ ,  $i = 1, 2$ . To do that we will use the results from Section 4.3. Equating with zero the remaining coefficients of  $\delta(x - \phi_i)$ ,  $i = 1, 2$ , (only such expressions  $(\text{mod } O_{D'}(\varepsilon))$  are left on the right-hand side of (22)), we obtain

$$\begin{aligned} \phi_{1t}(U(x_0(\phi_1, t, \tau), \rho) - u_0^0) - B_2(\rho)(f(U_1(x_0(\phi_1, t, \tau), \rho)) - f(u_0^0)) \\ - B_1(\rho)(f(U) - f(U_1(\xi(x_0(\phi_1, t, \tau)), \rho))) = 0, \end{aligned} \quad (32)$$

$$\begin{aligned} \phi_{2t}(U - U_1(x_0(\phi_2, t, \tau), \rho)) - B_2(\rho)(f(U) - f(U_1(x_0(\phi_2, t, \tau), \rho))) \\ - B_1(\rho)(f(U_1(\xi(x_0(\phi_2, t, \tau)), \rho)) - f(u_0^0)) = 0. \end{aligned} \quad (33)$$

According to (47) we have to prove that preceding equations are correct when  $\tau \rightarrow \pm\infty$  and to find  $\hat{\phi}$  such that their sum be equal to zero.

By the definition of the functions  $\phi_i(t, \tau)$ ,  $i = 1, 2$ , as  $\tau \rightarrow \infty$  (i.e., before the interaction), the limit of the expressions on the left-hand side of relations (29), (30) is equal to zero, and these relations admit the estimate  $O(\tau^{-N})$  for any  $N > 0$  as  $\tau \rightarrow \infty$ . This follows from the relations:  $\rho/\tau \rightarrow 1$  as  $\tau \rightarrow \infty$  and

$$B_2 = 1 + O(\rho^{-N}), \quad B_1 = O(\rho^{-N}), \quad \rho \rightarrow \infty \quad (\tau \rightarrow \infty). \quad (34)$$

We write the limit of these relations for  $\tau \rightarrow -\infty$ . Recall that

$$B_i(\rho) = \frac{1}{2} + O(|\tau|^{-1}), \quad \rho = \rho_0 + O(|\tau|^{-1}), \quad \tau \rightarrow -\infty, \quad i = 1, 2. \quad (35)$$

Therefore, denoting the limit of  $\phi_{it}$  as  $\tau \rightarrow -\infty$  by  $\phi_{it}^-$ ,  $i = 1, 2$ , we obtain

$$\phi_{it}^-\left(\frac{U - u_0^0}{2}\right) = \frac{1}{2}(f(U) - f(u_0^0)), \quad i = 1, 2, \quad (36)$$

or

$$\phi_{1t}^- = \frac{f(U) - f(u_0^0)}{U - u_0^0} = \phi_{2t}^-. \quad (37)$$

Denoting, as usual,

$$\frac{f(U) - f(u_0^0)}{U - u_0^0} = \frac{[f]}{[u]},$$

we can determine the general limit  $\phi^-(t)$  of the functions  $\phi_i(\tau, t)$ ,  $i = 1, 2$ , as  $\tau \rightarrow -\infty$  by the relation

$$\phi^- = \phi^-(t^*) + \frac{[f]}{[u]}(t - t^*). \quad (38)$$

Relations (36) (or (38)) mean that, for  $t > t^*$ , the trajectories  $x = \phi_1$  and  $x = \phi_2$  are close to the line

$$x - x^* = \frac{f(U) - f(u_0^0)}{U - u_0^0}(t - t^*),$$

i.e., to the trajectory of the shock wave (5).

Let us investigate the trajectories  $x = \phi_i$ ,  $i = 1, 2$ , in more detail.

By  $\omega(z)$  we denote the function satisfying the same conditions as the functions  $\omega_i$ ,  $i = 1, 2$ , in (10).

We prove that the following relations hold:

$$\phi_i(t, \tau) - \check{\phi}_i(t, \tau) = O(\varepsilon), \quad i = 1, 2, \quad (39)$$

where

$$\check{\phi}_i(t, \tau) = (1 - \omega(\tau))\hat{\phi}_i(\tau, t) + \omega(\tau)\left(x^* + \frac{[f]}{[u]}(t - t^*)\right).$$

Here  $\hat{\phi}_i(\tau, t) = X(a_i, t) + \psi_0 X_1(a_i, \tau)$ , (see (18)),  $x^* = \varphi_{10}(t^*) = \varphi_{20}(t^*)$ , and  $\phi_i(t, \tau)$ ,  $i = 1, 2$ , are the desired trajectories of singularities determined ( $\text{mod } O(\varepsilon)$ ) by the relations

$$\phi_i(t, \tau) = \hat{\phi}_i(\tau, t) + \psi_0 \hat{\phi}.$$

To prove (39), it suffices to set  $\phi^-(t^*) = x^*$  in (38) and to note that  $\hat{\phi}_i(0, t^*) = x^*$ ,  $i = 1, 2$ . It remains to note that the functions  $\hat{\phi}_i(\tau, t)$  can be represented in the form

$$\hat{\phi}_i(\tau, t) = x^* + \psi_0(\psi'_0 \tau)^{-1} \int_0^\tau [B_2 f'(U(a_i, \rho)) + B_1 f'(U(\xi(a_i), \rho)) + q(\tau', \rho)] d\tau'. \quad (40)$$

This follows from (18), (19) with the relation  $x^* = bt^* = -b\psi_0^0/\psi'_0$  taken into account (see formula (3)).

Now we apply Lemma 4.2 and see that relation (39) is proved. The statement we have proved means that  $\check{\phi}_i$  from (39) provide a family of expressions for trajectories close to those trajectories we want to construct. These approximate trajectories, with accuracy  $O(\varepsilon)$ , are independent of the choice of the function  $\omega(\tau)$ . It is only required that this function satisfy same conditions as the functions  $\omega_i$ ,  $i = 1, 2$ , from (10).

Now let us calculate the function  $x_0(\phi_i, t, \tau)$ ,  $i = 1, 2$ . By definition, this is the initial point of the trajectory  $x = \phi_i(t, \tau)$ ,  $i = 1, 2$ . Clearly, for  $t < t^*$ , we have  $\phi_i(t, \tau) = \hat{\phi}_i(t, \tau) + O(\varepsilon)$  and  $x_0(\phi_i, t, \tau) = a_i$ . For  $t > t^*$ , we have

$\phi_i(t, \tau) - \phi^-(t) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . By relation (37), for  $\phi^-(t^*) = x^*$ , we see that in this case the initial point is

$$\ddot{x} = \phi^-(0) = x^* - \frac{[f]}{[u]} t^*.$$

By the inequalities  $f'(U) < [f]/[u] < f'(u_0^0)$ , this implies that  $\ddot{x} \in (a_2, a_1)$ .

We set

$$\hat{X}_0(\phi_i, \tau) = a_i + \Omega(\tau)(\ddot{x} - a_i), \quad i = 1, 2,$$

where  $\Omega(\tau)$  is some (generating) function satisfying same conditions as the functions  $\omega_i$ ,  $i = 1, 2$ , from (10).

Let us prove the relations

$$\phi_i(t, \tau) - (x(\hat{X}_0(\phi_i, \tau), t, \tau) + \psi_0 \hat{\phi}) = O(\varepsilon), \quad i = 1, 2. \quad (41)$$

We restrict ourselves only to the case  $i = 1$ . We have

$$\begin{aligned} U_1(a_1, \rho) &= U_1(\hat{X}_0, \rho) - \Omega(\ddot{x} - a_1) \frac{\partial U_1}{\partial x_0}(a_1 + \alpha \Omega(\ddot{x} - a_1), \rho), \\ U_1(\xi(a_1), \rho) &= U + u_0^0 - U_1(a_1, \rho) \\ &= U + u_0^0 - U_1(\hat{X}_0, \rho) - (\ddot{x} - a_1) \frac{\partial U_1}{\partial x_0}(a_1 + \alpha \Omega(\ddot{x} - a_1), \rho), \end{aligned}$$

where  $\alpha \in (0, 1)$ .

From these relations, formula (40), and representation for  $\hat{x}$  from Section 2, we obtain

$$\begin{aligned} \hat{\varphi}_1(t, \tau) - \hat{x}(\hat{X}_0(\phi_1, \tau), t, \tau) &= (x - a_1) \psi_0(\psi'_0 \tau)^{-1} \int_0^\tau \left[ B_2 \frac{\delta f'}{\delta u}(U_1(a_1, \rho); U_1(\hat{X}_0, \rho)) \right. \\ &\quad \left. - B_1 \frac{\delta f'}{\delta u}(U + u_0^0 - U_1(a_1, \rho); U + u_0^0 - U_1(\hat{X}_0, \rho)) \right] \\ &\quad \times \Omega \frac{\partial U_1}{\partial x_0}(a_1 + \alpha \Omega(\ddot{x} - a_1), \rho) d\tau' \end{aligned} \quad (42)$$

where

$$\frac{\delta f'}{\delta u}(A, B) = \frac{f'(A) - f'(B)}{A - B} \xrightarrow[A \rightarrow B]{} f''(A).$$

We note that the integral on the right-hand side of (42) converges as  $\tau \rightarrow +\infty$  because the function  $\Omega$  is contained in the integrand. The convergence of the integral as  $\tau \rightarrow -\infty$  can be verified in the same way as the convergence of the last integral on the right-hand side of (20). Hence, by Lemma 4.2, we have

$$\hat{\varphi}_1(t, \tau) - x(\hat{X}_0(\phi_1, \tau), t, \tau) = O(\varepsilon),$$

and hence, by (38) and (39), we obtain (41). From (41) we obtain the relation

$$U_1(x_0(\phi_i, t, \tau), \rho) - U_1(\hat{X}_0(\phi_i, \tau), \rho) = O(\varepsilon), \quad i = 1, 2. \quad (43)$$

By construction, the limits of the expressions on the left-hand sides in (31) and (32) are equal to zero as  $\tau \rightarrow \infty$  (i.e., before the interaction). Moreover, the difference between the limit and the prelimit expression is  $O(\rho^{-N}) = O(\tau^{-N})$  for any  $N > 0$ .

By (32), these expressions also tend to zero as  $\tau \rightarrow -\infty$ , and the difference between the limit and the prelimit expression is  $O(B_1 - 1/2) = O(\rho - \rho_0) = O(|\tau|^{-1})$ ,  $\tau \rightarrow -\infty$ . Therefore, by the results of Sec. 4.2 about the linear independence, for the sum of terms with  $\delta$ -functions in (22) to admit the estimate  $O_{D'}(\varepsilon)$ , it is sufficient that the sum of expressions on the left-hand sides of (31) and (32) be equal to zero. Thus we obtain the equation

$$\begin{aligned} & \phi_{2t}(U - U_{1(2)}) + \phi_{1t}(U_{1(1)} - u_0^0) \\ &= B_2(\rho)(f(U_{1(1)}) - f(u_0^0)) + B_1(\rho)(f(U) - f(\hat{U}_{1(1)})) \\ &+ B_2(\rho)(f(U) - f(U_{1(2)})) + B_1(\rho)(f(\hat{U}_{1(2)}) - f(u_0^0)). \end{aligned} \quad (44)$$

Here, for brevity, we denote

$$U_{1(i)} = U_1(x_0(\phi_i, t, \tau), \rho), \quad \hat{U}_{1(i)} = U_1(\xi(x_0(\phi_i, t, \tau)), \rho), \quad i = 1, 2.$$

We note that

$$\phi_{it} = \hat{\phi}_{it} + \psi'_0 \frac{d}{d\tau}(\tau \hat{\phi}), \quad i = 1, 2,$$

We agree to denote  $f \approx g$  if

$$\lim \frac{f}{g} = 1.$$

It is easy to verify that as  $\tau \rightarrow \infty$ , we have

$$U - U_{1(2)} \approx U_{1(1)} - u_0^0 \approx U - \hat{U}_{1(1)} \approx \hat{U}_{1(2)} - u_0^0 \approx B_1(U - u_0^0). \quad (45)$$

Similarly,

$$\begin{aligned} f(U_{1(1)}) - f(u_0^0) &\approx f'(u_0^0)B_1(U - u_0^0), \\ f(U) - f(U_{1(2)}) &\approx f'(U)B_1(U - u_0^0), \\ f(U) - f(U_{1(1)}) &\approx f'(U)B_1(U - u_0^0), \\ f(\hat{U}_{1(2)}) - f(u_0^0) &\approx f'(u_0^0)B_1(U - u_0^0). \end{aligned} \quad (46)$$

Next, by (16), we have  $B'_2(\rho) \sim 1 - B_2$  and hence the relation  $g \sim 1 - B_2$  holds as  $\tau \rightarrow +\infty$ .

As  $\tau \rightarrow -\infty$ , the coefficient of  $\frac{d}{d\tau}(\tau \hat{\phi})$  in Eq. (44) is equal to  $U - u_0^0 \neq 0$ . Therefore, Eq. (44) is solvable for  $\hat{\phi}$  and its solution is a bounded function decreasing as  $\tau \rightarrow -\infty$ .

To write the solution of Eq. (44), we note that, with accuracy  $O(\varepsilon)$ , by (41), we can replace the arguments  $x_0(\phi_i, t, \tau)$  by  $X_0(\phi_i, \tau)$  in the functions  $U_{i(j)}$ , and by (38), the function  $X_0(\phi_i, \tau)$  can be determined actually independent of

the functions  $\phi_i$  (everywhere here  $i, j = 1, 2$ ). Hence Eq. (44) is indeed a linear equation with respect to  $\hat{\phi}$  and its solution can be easily found.

This solution has the form

$$\begin{aligned}\hat{\phi} &= (\psi'_0 \tau)^{-1} \int_0^\tau \left( U - u_0^0 - U_1(\hat{X}_0(\phi_2, \tau), \rho) + U_1(\hat{X}_0(\phi_1, \tau), \rho) \right)^{-1} \\ &\quad \times \left( -\hat{\varphi}_2[U - U_1(\hat{X}_0(\phi_2, \tau), \rho)] - \hat{\varphi}_1[U_1(\hat{X}_0(\phi_1, \tau), \rho) - u_0^0] \right. \\ &\quad \left. + \{B_2(\rho)(f(U_{1(1)}) - f(u_0^0)) + B_1(\rho)(f(U) - f(\hat{U}_{1(1)})) \right. \\ &\quad \left. + B_2(\rho)(f(U) - f(U_{1(2)})) + B_1(\rho)(f(\hat{U}_{1(2)}) - f(u_0^0))\} \right) d\tau'.\end{aligned}$$

By (45) and (46), the integral on the right-hand side in the last relation converges as  $\tau \rightarrow \infty$  and  $\hat{\phi} = O(\tau^{-1})$  as  $\tau \rightarrow \infty$ .

## 4 Auxiliary formulas and statements of weak asymptotic method

### 4.1 Nonlinear superposition of approximations of Heaviside functions

Suppose that  $\omega_j(z) \rightarrow 0, 1$  as  $z \rightarrow -\infty, z \rightarrow \infty$ ,  $\frac{d^\alpha \omega_j}{dz^\alpha} = O(|z|^{-N})$ ,  $j = 1, 2$ ,  $|z| \rightarrow \infty$ ,  $N$  is a sufficiently large number, and  $\varphi_1, \varphi_2$  are some continuous functions of the variable  $t$ .

It is easy to verify that the functions  $\omega_j((x - \phi(t))/\varepsilon)$  approximate in the weak sense the Heaviside function  $H(x - \phi(t))$ . Indeed, the properties of the functions  $\omega_j(z)$  imply the relations

$$\omega_j(z) - H(z) = O(|z|^{-N}), \quad N > 0, \quad j = 1, 2.$$

Hence, for any test function  $\psi(x)$ , we have

$$\left\langle \omega_j\left(\frac{x - \phi}{\varepsilon}\right) - H(x - \varphi), \psi \right\rangle = \varepsilon \int (\omega_j(z) - H(z)) \psi(\phi + \varepsilon z) dz = O(\varepsilon).$$

**Lemma 4.1.** *For any  $C^1$ -function  $f(x)$ , the following relation holds:*

$$\begin{aligned}f\left(a + b\omega_1\left(\frac{\varphi_1 - x}{\varepsilon}\right) + b\omega_2\left(\frac{\varphi_2 - x}{\varepsilon}\right)\right) \\ = f(a) + H(\varphi_1 - x)\{B_2(f(a+b) - f(a)) + B_1(f(a+b+c) - f(a+c))\} \\ + H(\varphi_2 - x)\{B_1(f(a+c) - f(a)) + B_2(f(a+b+c) - f(a+b))\} + O_{D'}(\varepsilon),\end{aligned}$$

where

$$\begin{aligned}B_j = B_j\left(\frac{\varphi_1 - \varphi_2}{\varepsilon}\right), \quad j = 1, 2 \quad B_1 + B_2 = 1, \\ B_2(z) \rightarrow 1 \quad \text{as } z \rightarrow \infty, \quad B_2(z) \rightarrow 0 \quad \text{as } z \rightarrow -\infty.\end{aligned}$$

**Proof:** First, we prove the relation

$$f\left(a+b\omega_1\left(\frac{\varphi_1-x}{\varepsilon}\right)+c\omega_2\left(\frac{\varphi_2-x}{\varepsilon}\right)\right) = f(a+bH(\varphi_1-x)+cH(\varphi_2-x))+O_{\mathcal{D}'}(\varepsilon).$$

Indeed, we have

$$\begin{aligned} f(a+b\omega_1+c\omega_2) &= f\left(a+bH(\varphi_1-x)+c(\varphi_2-x)\right) \\ &\quad + f'\left(a+\xi(b(\omega_1-H(\varphi_1-x)))+c(\omega_2-H(\varphi_2-x))\right) \\ &\quad \times [(\omega_1-H(\varphi_1-x))b+(\omega_2-H(\varphi_2-x))c]. \end{aligned}$$

Now we verify that if  $g(x, \varphi, \varepsilon)$  is a bounded function, then

$$g(x, \varphi, \varepsilon)[\omega_1 - H(\varphi_1 - x)] = O_{\mathcal{D}'}(\varepsilon).$$

For any test function  $\psi(x)$ , we have

$$\begin{aligned} &\left| \int g(x, \varphi, \varepsilon) \left[ \omega\left(\frac{\varphi_1-x}{\varepsilon}\right) - H(\varphi_1-x) \right] \psi(x) dx \right| \\ &= \left| \varepsilon \int g(\varphi_1 + \varepsilon z, \varphi, \varepsilon) [\omega(z) - H(z)] \psi(\varphi - \varepsilon z) dz \right| \\ &\leq \varepsilon \text{const} \int |\omega(z) - H(z)| dz. \end{aligned}$$

This implies

$$f(a+b\omega_1+c\omega_2) = f(a+bH(\varphi_1-x)+cH(\varphi_2-x))+O_{\mathcal{D}'}(\varepsilon).$$

Next, it is easy to verify the relation

$$\begin{aligned} f(a+bH_1+cH_2) &= f(a) + H_1[f(a+b)-f(a)] + H_2[f(a+c)-f(a)] \\ &\quad + H_1H_2(f(a+b+c)-f(a+c)-f(a+b)+f(a)), \\ H_j &\stackrel{\text{def}}{=} H(\varphi_j-x), \quad j=1,2. \end{aligned}$$

It remains to note that we have

$$\begin{aligned} H(\varphi_1-x)H(\varphi_2-x) &= B_1H(\varphi_1-x)+B_2H(\varphi_2-x)+O_{\mathcal{D}'}(\varepsilon), \\ B_1 &= \int \dot{\omega}_1(z)\omega_2\left(z-\frac{\varphi_1-\varphi_2}{\varepsilon}\right) dz, \quad B_2 = 1-B_1. \end{aligned}$$

For the proof of these and similar relations, see [1, 2, 7]. The proof of the lemma is complete.  $\square$

## 4.2 Asymptotic linear independence

If we want to consider linear combinations of generalized functions with accuracy  $O_{\mathcal{D}'}(\varepsilon^\alpha)$ , then we need to modify the notion of linear independence. This modification plays the key role in considerations related to the soliton interaction problem.

Indeed, let  $\phi_1 \neq \phi_2$  be independent of  $x$ . We consider the relation

$$g_1\delta(x - \phi_1) + g_2\delta(x - \phi_2) = O_{\mathcal{D}'}(\varepsilon^\alpha), \quad \alpha > 0,$$

where  $g_i$  are independent of  $\varepsilon$ . Obviously, we obtain the relations

$$g_i = O_{\mathcal{D}'}(\varepsilon^\alpha), \quad i = 1, 2,$$

which, by our assumption, imply

$$g_i = 0, \quad i = 1, 2.$$

Everything is different [1] if we assume that the coefficients  $g_i$ ,  $i = 1, 2$ , can depend on  $\varepsilon$ . Here we consider only a special case of such dependence, which we shall use later. Namely, let

$$g_i = A_i + S_i(\Delta\phi/\varepsilon), \quad i = 1, 2,$$

where  $A_i$  are independent of  $\varepsilon$  and  $S_i(\rho)$  decrease as  $|\rho| \rightarrow \infty$ .

We assume that the estimate holds:

$$|\rho S_i(\rho)| \leq \text{const}, \quad i = 1, 2.$$

Let us find out what properties of the coefficients  $g_i$  follow from the relation

$$g_1\delta(x - \phi_1) + g_2\delta(x - \phi_2) = O_{\mathcal{D}'}(\varepsilon).$$

Applying both sides of the equality to a test function  $\varphi$ , we obtain

$$g_1\varphi(\phi_1) + g_2\varphi(\phi_2) = O(\varepsilon)$$

or, which is the same,

$$[A_1\varphi(\phi_1) + A_2\varphi(\phi_2)] + [S_1\varphi(\phi_1) + S_2\varphi(\phi_2)] = O(\varepsilon). \quad (47)$$

Let us consider the expression in the second brackets. Using Taylor's formula, we obtain

$$[S_1\varphi(\phi_1) + S_2\varphi(\phi_2)] = S_1\varphi(\phi_1) + S_2\varphi(\phi_1) + S_2(\phi_2 - \phi_1)\varphi'(\phi_1 + \theta\phi_2), \quad 0 < \theta < 1.$$

Now we see that

$$S_2(\Delta\phi/\varepsilon)(\phi_2 - \phi_1) = \{-\rho S_2(\rho)\}|_{\rho=\Delta\phi/\varepsilon} \cdot \varepsilon = O(\varepsilon),$$

since the function  $\rho S_2(\rho)$  is bounded uniformly in  $\rho \in \mathbb{R}^1$ .

So we can rewrite relation (37) as

$$A_1\varphi(\phi_1) + A_2\varphi(\phi_2) + (S_1 + S_2)\varphi(\phi_1) = O(\varepsilon).$$

Hence, as the coefficients  $A_i$  are independent of  $\varepsilon$ , we, as usual, obtain

$$A_1 = 0, \quad A_2 = 0, \quad S_1 + S_2 = 0. \quad (48)$$

Another method for analyzing relation (37) is the following. We assume that  $\phi_i(t)$  are smooth functions, the relation  $\phi_1(t^*) = \phi_2(t^*)$  holds for some  $t = t^*$ , and, moreover,  $\phi'_1(t^*) \neq \phi'_2(t^*)$ . Then

$$\begin{aligned} \langle S_1\delta(x - \phi_1), \varphi \rangle + \langle S_2\delta(x - \phi_2), \varphi \rangle &= S_1\varphi(x^*) + S_2\varphi(x^*) \\ &+ S_1O(t - t^*) + S_2O(t - t^*), \quad x^* = \phi_1(t^*) = \phi_2(t^*). \end{aligned}$$

But  $O(t - t^*) \sim O(\Delta\phi)$ . Therefore, we have

$$\begin{aligned} (A_1 + S_1)\delta(x - \phi_1) + (A_2 + S_2)\delta(x - \phi_2) \\ = A_1\delta(x - \phi_1) + A_2\delta(x - \phi_1) + (S_1 + S_2)\delta(x - x^*) + O_{D'}(\varepsilon). \end{aligned}$$

We again obtain relations (47).

### 4.3 Complex germ lemma

In this section, in the form convenient for us, we present the statement that plays an important role in Maslov's complex germ theory [8, 9].

**Lemma 4.2.** *Let  $f(t) \in C^1$ ,  $f(t_0) = 0$ , and  $f'(t_0) \neq 0$ . Let  $g(\tau, t)$  be a function that locally uniformly satisfies the estimates*

$$|\tau g(\tau, t)| \leq \text{const}, \quad |\tau g'_t(\tau, t)| \leq \text{const}, \quad -\infty < \tau < \infty,$$

and  $g(\tau, t_0) = 0$ . Then the inequality

$$\left| g\left(\frac{f(t)}{\varepsilon}, t\right) \right| \leq C_T \varepsilon,$$

where  $C_T = \text{const}$ , holds on any interval  $0 \leq t \leq T$  that does not contain zeros of the function  $f(t)$  except  $t_0$ .

**Proof:** The fraction  $f(t)/(t - t_0)$  is locally bounded in  $t$ . The fraction  $\tau g(\tau, t)/(t - t_0)$  is also locally bounded. We have

$$g\left(\frac{f(t)}{\varepsilon}, t\right) = \varepsilon \left[ g\left(\frac{f(t)}{\varepsilon}, t\right) (t - t_0)^{-1} \right] \frac{f(t)}{\varepsilon} \cdot \frac{t - t_0}{f(t)}.$$

By the assumptions of the lemma, on the interval under study, the last multiplier on the right-hand side is bounded, while the product of the intermediate multiplier and the expression in square brackets is bounded in view of the properties of the function  $g(\tau, t)$ .  $\square$

**Corollary 4.3.** Suppose that the estimates in the assumptions of the lemma hold for  $0 \leq \tau < \infty$  ( $-\infty < \tau \leq 0$ ). Then the statement of the lemma holds on any interval  $[t_0, T]$  that does not contain zeros of the function  $f(t)$  and  $\text{sgn}T = \text{sgn}f(t)$ ,  $t \in [t_0, T]$ .

## 5 Justification of the weak asymptotic solution

In this section we will prove that our weak asymptotic solution is in some sense "close" to the admissible weak solution of problem (1), (2).

The existence of the admissible weak solution in our situation is obvious by Kruzhkov theorem (see [5], Chapter 6).

We will introduce admissibility conditions necessary for the uniqueness of the weak solution of considered problem.

**Definition 5.1.** (Oleinik admissibility condition) We say that a weak solution  $u(t, x)$ ,  $t \in \mathbb{R}^+$ ,  $x \in \mathbb{R}$ , of problem (1), (2) is admissible if it satisfies

$$u_+ = u(t, x^* + 0) < u(t, x^* - 0) = u_-.$$

in every point of its discontinuity.

Notice that such condition we can use only when the function  $u$  is piecewise continuous weak solutions of the considered problem for every fixed  $t \in \mathbb{R}^+$ . In that case Definition 5.1 is equivalent to more general Kruzhkov admissibility condition (which can be applied on functions which are merely measurable):

**Definition 5.2.** We say that the weak solution  $u(x, t)$ ,  $x \in \mathbb{R}$ ,  $t \in \mathbb{R}^+$  of problem (1), (2) is admissible if we have

$$\int_0^T \int_{\mathbb{R}} [\partial_t \psi \eta(u) + \partial_x \psi q(u)] dx dt + \int_{\mathbb{R}} \psi(x, 0) \eta(u_0(x)) dx \geq 0, \quad (49)$$

where  $q(u) = \int \eta'(u) f'(u) du$  and  $\eta \in C^1(\mathbb{R})$  is an arbitrary convex function.

Using this definition, Kruzhkov proved the existance uniqueness theorem (i.e. Theorem 6.2.2 in [5]).

We will prove that weak asymptotic solution tends in  $L^1$  to the admissible weak solution of problem (1), (2). By the definition of the weak asymptotic solution for all  $\varphi \in C^\infty([0, T]; C_0^\infty(\mathbb{R}^1))$  we have:

$$\int_{\mathbb{R}} [u_{\varepsilon t} + (f(u_\varepsilon))_x] \phi(x, t) dx = \mathcal{O}(\varepsilon),$$

uniformly in  $t \in [0, T]$ ,  $T \in \mathbb{R}^+$ . Integrating last expression with  $\int_0^T dt$  we obtain:

$$\int_0^T \int_{\mathbb{R}} [u_{\varepsilon t} + (f(u_\varepsilon))_x] \phi(x, t) dx dt = \mathcal{O}(\varepsilon). \quad (50)$$

Now letting  $\varepsilon \rightarrow 0$  we see that  $u(x, t) = w - \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t)$  is the weak solution of (1), (2). From the construction we see that  $u$  satisfy Oleinik admissibility condition (since  $u$  is obviously piecewise continuous) and this implies Kruzhkov admissibility condition. Furthermore, it is easy to see that we have:

$$\int_0^T \int_{\mathbb{R}} [\partial_t \psi \eta(u_\varepsilon) + \partial_x \psi q(u_\varepsilon)] dx dt + \int_{\mathbb{R}} \psi(x, 0) \eta(u_0(x)) dx \geq \varepsilon \mathcal{O}(1), \quad (51)$$

where  $q(u) = \int \eta'(u) f'(u) du$  and  $\eta \in C^1(\mathbb{R})$  is an arbitrary convex function.

Relation (50) holds by (49) and the smoothness of the function  $u_\varepsilon(x, t)$  for  $\varepsilon > 0$ .

Now we can repeat the procedure from [5], Theorem 6.2.2, page 87., to obtain:

**Theorem 5.3.** *Let  $u_\varepsilon$  and  $u$  satisfy (48) and (50), respectively. There exists  $s > 0$  depending only on  $[u_0^0, U]$  (interval in which initial data take values) such that for every  $t \in [0, T)$  and every  $r > 0$  we have:*

$$\|u(\cdot, t) - u_\varepsilon(\cdot, t)\|_{L^1(|x| < r)} \leq (r + st) \cdot \varepsilon \mathcal{O}(1).$$

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